

# EXPLICIT FORMULAS FOR NON-GEODESIC BIHARMONIC CURVES OF THE HEISENBERG GROUP

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ABSTRACT. We consider the biharmonicity condition for maps between Riemannian manifolds (see [BK]), and study the non-geodesic biharmonic curves in the Heisenberg group  $\mathbb{H}_3$ . First we prove that all of them are helices, and then we obtain explicitly their parametric equations.

## 1. INTRODUCTION

By the definition given by J. Eells and J.H. Sampson in [ES], a map  $\phi$  from a compact Riemannian manifold  $(M, g)$  to another Riemannian manifold  $(\overline{M}, \overline{g})$  is harmonic if it is a critical point of the energy

$$E_1(\phi) = \frac{1}{2} \int_M |d\phi|^2 v_g.$$

From the first variation formula it follows that  $\phi$  is harmonic if and only if its first tension field  $\tau_1(\phi) = \text{trace } \nabla d\phi$  vanishes. The same authors suggested a generalization of the notion of harmonicity: a map  $\phi$  is biharmonic if it is a critical point of the bienergy functional

$$E_2(\phi) = \frac{1}{2} \int_M |\tau_1(\phi)|^2 v_g.$$

The first variation formula for  $E_2$ , given by G.Y. Jiang in [Y1] and [Y2], is reobtained in [BK] and can be written as

$$\left. \frac{dE_2(\phi_t)}{dt} \right|_{t=0} = \int_M \langle \tau_2(\phi), V \rangle v_g,$$

where  $\{\phi_t\}_t$  is a variation of  $\phi$ ,  $V$  is the variational vector field along  $\phi$ , and

$$\begin{aligned} (1.1) \quad \tau_2(\phi) &= -J(\tau_1(\phi)) \\ &= -\Delta \tau_1(\phi) + \text{trace } R^{\overline{M}}(d\phi, \tau_1(\phi))d\phi. \end{aligned}$$

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Here  $J$  denotes the Jacobi operator and  $\Delta\tau_1(\phi) = -\text{trace}(\nabla^\phi)^2\tau_1(\phi)$ , where  $\nabla^\phi$  represents the connection in  $\phi^{-1}(T\overline{M})$ . Hence the condition of biharmonicity for  $\phi$  is

$$(1.2) \quad \tau_2(\phi) = 0.$$

When  $M$  is not compact, as a definition of biharmonicity one can adopt equation (1.2) instead of the definition given in terms of the integral formula of the bienergy.

Since any harmonic map is evidently biharmonic, we are interested in non-harmonic biharmonic maps.

In general, the biharmonic equation is very complicated, and therefore the problem of finding examples of non-harmonic biharmonic maps between two Riemannian manifolds, or examples of non-harmonic (non-minimal) biharmonic submanifolds of a given Riemannian manifold, is difficult to solve. Still there are some results that now we mention.

- Biharmonic curves on a surface in  $\mathbb{R}^3$  have been considered in [CMP1].
- In [CMO1] the authors gave the complete classification of non-minimal biharmonic submanifolds of  $\mathbb{S}^3$ . These are certain circles, spherical helices and parallel spheres. The general case of  $\mathbb{S}^n$ , for  $n > 3$ , is more interesting and has been treated in [CMO2].
- More recently, in [BK], it has been shown how to construct non-harmonic biharmonic maps from  $M$  to  $\overline{M}$  by deforming conformally the metric of  $M$ .
- When  $M$  is a submanifold of the Euclidean space  $\mathbb{R}^n$ , the biharmonicity condition seems to be very restrictive. In fact, B.Y. Chen conjectured in [CI] that any biharmonic submanifold of the Euclidean space  $\mathbb{R}^n$  is minimal. This was proved to be true in the case of submanifolds in  $\mathbb{R}^3$  (in [CI]) and in other special cases (see [D], [HV] and [C]).
- The same result was proved in [CMO2] for any 3-dimensional Riemannian manifold  $\overline{M}^3(-1)$  with constant negative sectional curvature  $-1$ .
- Other results about non-existence of non-harmonic biharmonic maps, when the sectional curvature or the Ricci curvature of  $\overline{M}$  is non-positive, can be found in [Y1], [O1] and [O2].

It seems then natural, as the next step, to consider biharmonic submanifolds in a 3-manifold  $\overline{M}$  with non-constant sectional curvature. We choose as  $\overline{M}$  the Heisenberg group  $\mathbb{H}_3$ . This nilpotent Lie group is not-symmetric; nevertheless it has many symmetries, in the sense that its isometry group has dimension four, the biggest possible for a 3-space of non-constant sectional curvature. It is probably in virtue of these symmetries that some main problem turns out to be easier than expected. As, for instance, the problem of finding the equations of geodesics (see [K] and [H]), and that of determining all surfaces with constant mean or Gaussian curvature invariant

with respect to some subgroup of the isometry group (see [T], [CPR1], [CPR2] and [FMP]). This manifold has many interesting properties. We mention, for example, that there are no totally umbilical surfaces and therefore there are no totally geodesic surfaces in  $\mathbb{H}_3$  (see [S]).

In this paper we first write down the conditions that any non-harmonic (non-geodesic) biharmonic curve in  $\mathbb{H}_3$  must satisfy. Then we prove that the non-geodesic biharmonic curves in  $\mathbb{H}_3$  are helices. A similar fact occurs in  $\mathbb{S}^3$ . Finally we deduce the explicit parametric equations of the non-geodesic biharmonic curves in  $\mathbb{H}_3$ .

*Notation.* We shall work in  $C^\infty$  category, i.e. manifolds, metrics, connections, maps will be assumed to be smooth.

## 2. RIEMANNIAN STRUCTURE OF $\mathbb{H}_3$

The Heisenberg group  $\mathbb{H}_3$  can be seen as the Euclidean space  $\mathbb{R}^3$  endowed with the multiplication

$$(\tilde{x}, \tilde{y}, \tilde{z})(x, y, z) = (\tilde{x} + x, \tilde{y} + y, \tilde{z} + z + \frac{1}{2}\tilde{x}y - \frac{1}{2}\tilde{y}x)$$

and with the Riemannian metric  $g$  given by

$$(2.1) \quad g = dx^2 + dy^2 + (dz + \frac{y}{2}dx - \frac{x}{2}dy)^2.$$

The metric  $g$  is invariant with respect to the left-translations corresponding to that multiplication. This metric is isometric to the other, also quite standard, which is left-invariant with respect to the composition arising from the multiplication of the  $3 \times 3$  Heisenberg matrices.

At each point the metric  $g$  has an axial symmetry: the 4-dimensional group of its isometries contains the group of rotations around the  $z$  axis (in the classical terminology (see [B]) a space with such a property is called *systatic*).

First of all we shall determine the Levi-Civita connection  $\nabla$  of the metric  $g$  with respect to the left-invariant orthonormal basis

$$e_1 = \frac{\partial}{\partial x} - \frac{1}{2}y \frac{\partial}{\partial z}, \quad e_2 = \frac{\partial}{\partial y} + \frac{1}{2}x \frac{\partial}{\partial z}, \quad e_3 = \frac{\partial}{\partial z},$$

which is dual to the coframe

$$\theta^1 = dx, \quad \theta^2 = dy, \quad \theta^3 = dz + \frac{y}{2}dx - \frac{x}{2}dy.$$

We obtain

$$(2.2) \quad \begin{cases} \nabla_{e_1} e_1 = 0, & \nabla_{e_1} e_2 = \frac{1}{2}e_3, & \nabla_{e_1} e_3 = -\frac{1}{2}e_2, \\ \nabla_{e_2} e_1 = -\frac{1}{2}e_3, & \nabla_{e_2} e_2 = 0, & \nabla_{e_2} e_3 = \frac{1}{2}e_1, \\ \nabla_{e_3} e_1 = -\frac{1}{2}e_2, & \nabla_{e_3} e_2 = \frac{1}{2}e_1, & \nabla_{e_3} e_3 = 0. \end{cases}$$

Also, we have the well known Heisenberg bracket relations

$$[e_1, e_2] = e_3, \quad [e_3, e_1] = [e_2, e_3] = 0.$$

We shall adopt the following notation and sign convention. The curvature operator is given by

$$R(X, Y)Z = -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X, Y]} Z,$$

while the Riemann-Christoffel tensor field and the Ricci tensor field are given by

$$R(X, Y, Z, W) = g(R(X, Y)Z, W), \quad \rho(X, Y) = \text{trace}(Z \rightarrow R(X, Z)Y),$$

where  $X, Y, Z, W$  are smooth vector fields on  $\mathbb{H}_3$ . Moreover we put, as in [PS] or [S],

$$R_{abc} = R(e_a, e_b)e_c, \quad R_{abcd} = R(e_a, e_b, e_c, e_d), \quad \rho_{ab} = \rho(e_a, e_b),$$

where the indices  $a, b, c, d$  take the values 1, 2, 3. The non vanishing components of the above tensor fields are

$$(2.3) \quad \begin{cases} R_{121} = -\frac{3}{4}e_2, & R_{131} = \frac{1}{4}e_3, & R_{122} = \frac{3}{4}e_1, & R_{232} = \frac{1}{4}e_3, \\ R_{133} = -\frac{1}{4}e_1, & R_{233} = -\frac{1}{4}e_2, \end{cases}$$

$$(2.4) \quad R_{1212} = -\frac{3}{4}, \quad R_{1313} = R_{2323} = \frac{1}{4},$$

$$(2.5) \quad \rho_{11} = \rho_{22} = -\frac{1}{2}, \quad \rho_{33} = \frac{1}{2},$$

respectively, and those obtained from these by means of the symmetries of  $R$ . Thus the curvatures of  $\mathbb{H}_3$  have both positive and negative components.

### 3. BIHARMONIC CURVES IN $\mathbb{H}_3$

To study the biharmonic curves in  $\mathbb{H}_3$ , we shall use their Frenet vector fields and equations. Let  $\gamma : I \rightarrow \mathbb{H}_3$  be a differentiable curve parametrized by arc length and let  $\{T, N, B\}$  be the orthonormal frame field tangent to  $\mathbb{H}_3$  along  $\gamma$  and defined as follows: by  $T$  we denote the unit vector field  $\gamma'$  tangent to  $\gamma$ , by  $N$  the unit vector field in the direction of  $\nabla_T T$  normal to  $\gamma$ , and we choose  $B$  so that  $\{T, N, B\}$  is a positive oriented orthonormal basis. Then we have the following Frenet equations

$$(3.1) \quad \begin{aligned} \nabla_T T &= kN \\ \nabla_T N &= -kT - \tau B, \\ \nabla_T B &= \tau N \end{aligned}$$

where  $k = |\tau_1(\gamma)| = |\nabla_T T|$  is the geodesic curvature of  $\gamma$  and  $\tau$  its geodesic torsion. By making use of equations (3.1) and of expression (2.3) of the curvature tensor field, we obtain from (1.1) the biharmonic equation for  $\gamma$ :

$$\begin{aligned} \tau_2(\gamma) &= \nabla_T^3 T + R(T, kN)T \\ &= (-3k'k)T + (k'' - k^3 - k\tau^2 + \frac{k}{4} - kB_3^2)N + (-2k'\tau - k\tau' + kN_3B_3)B \\ &= 0, \end{aligned}$$

where  $T = T_1e_1 + T_2e_2 + T_3e_3$ ,  $N = N_1e_1 + N_2e_2 + N_3e_3$ , and  $B = T \times N = B_1e_1 + B_2e_2 + B_3e_3$ . Thus we have

**Theorem 3.1.** *Let  $\gamma : I \rightarrow \mathbb{H}_3$  be a differentiable curve parametrized by arc length. Then  $\gamma$  is a non-geodesic biharmonic curve if and only if*

$$(3.2) \quad \begin{cases} k = \text{constant} \neq 0; \\ k^2 + \tau^2 = \frac{1}{4} - B_3^2; \\ \tau' = N_3B_3. \end{cases}$$

**Remark 3.2.** By analogy with curves in  $\mathbb{R}^3$ , also following [G], we keep the name *helix* for a curve in a Riemannian manifold having constant both geodesic curvature and geodesic torsion. Now, for any helix in  $\mathbb{H}_3$ , the system (3.2) becomes

$$(3.3) \quad \begin{cases} k^2 + \tau^2 = \frac{1}{4} - B_3^2; \\ N_3B_3 = 0, \end{cases}$$

and therefore, in this case,  $B_3$  must be constant, too.

Thus biharmonic helices satisfy

$$(3.4) \quad \begin{cases} B_3 = \text{constant}; \\ k^2 + \tau^2 = \frac{1}{4} - B_3^2; \\ N_3 B_3 = 0. \end{cases}$$

We shall come back to biharmonic helices in the next section, after showing that all the biharmonic curves in  $\mathbb{H}_3$  are helices. First we prove that for a biharmonic curve in  $\mathbb{H}_3$  the geodesic torsion must be constant.

**Proposition 3.3.** *Let  $\gamma : I \rightarrow \mathbb{H}_3$  be a non-geodesic curve parametrized by arc length. If  $k$  is constant and  $N_3 B_3 \neq 0$ , then  $\gamma$  is not biharmonic.*

*Proof.* From

$$\begin{aligned} \nabla_T T &= (T'_1 + T_2 T_3) e_1 + (T'_2 - T_1 T_3) e_2 + T'_3 e_3 \\ &= k N, \end{aligned}$$

we obtain  $T'_3 = k N_3$ ; then, if we put  $T_3(s) = k F(s)$  and  $f(s) = F'(s)$ , we get  $N_3(s) = f(s)$ . Hence we write

$$T = \sqrt{1 - k^2 F^2} \cos \beta(s) e_1 + \sqrt{1 - k^2 F^2} \sin \beta(s) e_2 + k F(s) e_3$$

and then we use the first Frenet equation, that gives

$$\begin{aligned} \nabla_T T &= \left( \sqrt{1 - k^2 F^2} (k F - \beta') \sin \beta - \frac{k^2 F f}{\sqrt{1 - k^2 F^2}} \cos \beta \right) e_1 \\ &\quad - \left( \sqrt{1 - k^2 F^2} (k F - \beta') \cos \beta + \frac{k^2 F f}{\sqrt{1 - k^2 F^2}} \sin \beta \right) e_2 + k f e_3 \\ &= k N. \end{aligned}$$

Since  $k^2 = |\nabla_T T|^2$ , we have

$$k F - \beta' = \pm k \frac{\sqrt{1 - f^2 - k^2 F^2}}{1 - k^2 F^2}.$$

Now we replace  $k F - \beta'$  in the above expression of  $\nabla_T T$ , and we obtain

$$\begin{aligned}
N = & \left( \pm \frac{\sqrt{1-f^2-k^2F^2}}{\sqrt{1-k^2F^2}} \sin \beta - \frac{kFf}{\sqrt{1-k^2F^2}} \cos \beta \right) e_1 \\
& + \left( \mp \frac{\sqrt{1-f^2-k^2F^2}}{\sqrt{1-k^2F^2}} \cos \beta - \frac{kFf}{\sqrt{1-k^2F^2}} \sin \beta \right) e_2 + f e_3.
\end{aligned}$$

As  $B = T \times N$ , we have  $B_3 = T_1N_2 - N_1T_2 = \mp \sqrt{1-f^2-k^2F^2}$ . Then the second Frenet equation gives

$$(3.5) \quad \langle \nabla_T N, e_3 \rangle = \langle -kT - \tau B, e_3 \rangle = -kT_3 - \tau B_3.$$

On the other hand we have

$$\begin{aligned}
\langle \nabla_T N, e_3 \rangle &= \langle \nabla_T (N_1 e_1 + N_2 e_2 + N_3 e_3), e_3 \rangle \\
&= \langle (N'_1 + \frac{1}{2}(T_2 N_3 + T_3 N_2)) e_1 + (N'_2 - \frac{1}{2}(T_1 N_3 + T_3 N_1)) e_2 \\
&\quad + (N'_3 + \frac{1}{2}(T_1 N_2 - N_1 T_2)) e_3, e_3 \rangle \\
(3.6) \quad &= N'_3 + \frac{1}{2} B_3.
\end{aligned}$$

By comparing (3.5) and (3.6) we obtain

$$(3.7) \quad N'_3 + \frac{B_3}{2} = -kT_3 - \tau B_3.$$

Next we replace  $N_3 = f$ ,  $B_3 = \mp \sqrt{1-f^2-k^2F^2}$  and  $T_3 = kF$  in (3.7), and we get

$$(3.8) \quad \tau = \pm \frac{f' + k^2 F}{\sqrt{1-f^2-k^2F^2}} - \frac{1}{2} = \frac{B'_3}{N_3} - \frac{1}{2}.$$

Assume now that  $\gamma$  is biharmonic. Then  $\tau' = N_3 B_3 \neq 0$  and we can write

$$N_3 = \frac{\tau'}{B_3}.$$

By substituting  $N_3$  in (3.8) and then by integrating we get

$$(3.9) \quad \tau^2 = B_3^2 - \tau + c,$$

where  $c$  is a constant. On the other hand, from the second equation in (3.2), we

obtain  $B_3^2 = \frac{1}{4} - k^2 - \tau^2$ . Thus equation (3.9) becomes

$$2\tau^2 + \tau = C,$$

where  $C$  is a constant, and therefore also  $\tau$  is constant, and we have a contradiction.  $\square$

From Theorem 3.1 and Proposition 3.3 we have, in conclusion,

**Theorem 3.4.** *Let  $\gamma : I \rightarrow \mathbb{H}_3$  be a non-geodesic curve parametrized by arc length. Then  $\gamma$  is biharmonic if and only if*

$$(3.10) \quad \begin{cases} k = \text{constant} \neq 0; \\ \tau = \text{constant}; \\ N_3 B_3 = 0; \\ k^2 + \tau^2 = \frac{1}{4} - B_3^2. \end{cases}$$

#### 4. BIHARMONIC HELICES IN $\mathbb{H}_3$

Now we want to determine all helices in  $\mathbb{H}_3$  that are biharmonic but non-geodesic. From Theorem 3.4 it is clear that, to this aim, we have to study the behaviour of  $N_3$  and  $B_3$ .

For one thing, it follows from (3.10) that  $B_3$  must be constant. We shall show that for a curve  $\gamma$  satisfying (3.10) the constant  $B_3$  cannot vanish. More precisely we prove

**Proposition 4.1.** *Let  $\gamma : I \rightarrow \mathbb{H}_3$  be a non-geodesic curve parametrized by arc length. If  $B_3 = 0$ , then  $\tau^2 = \frac{1}{4}$  and  $\gamma$  is not biharmonic.*

*Proof.* As  $\gamma$  is parametrized by arc length, we can write

$$T = \sin \alpha \cos \beta e_1 + \sin \alpha \sin \beta e_2 + \cos \alpha e_3,$$

where  $\alpha = \alpha(s)$ ,  $\beta = \beta(s)$ . By using (2.2) we first have

$$\begin{aligned} \nabla_T T &= (\alpha' \cos \alpha \cos \beta - \sin \alpha \sin \beta (\beta' - \cos \alpha)) e_1 + \\ &+ (\alpha' \cos \alpha \sin \beta + \sin \alpha \cos \beta (\beta' - \cos \alpha)) e_2 - \alpha' \sin \alpha e_3 \\ &= kN. \end{aligned}$$

Next we compute  $B = T \times N$ , and obtain

$$B_3 = \frac{\sin^2 \alpha (\beta' - \cos \alpha)}{k}.$$

Assume now  $B_3 = 0$ . We exclude the case  $\sin \alpha = 0$ , that implies  $T = e_3$  and therefore that  $\gamma$  is a geodesic. Thus we must have  $\beta' - \cos \alpha = 0$ , and hence

$$\nabla_T T = \alpha'(\cos \alpha \cos \beta e_1 + \cos \alpha \sin \beta e_2 - \sin \alpha e_3).$$

Without loss of generality, we can assume that  $\alpha' > 0$  (when  $\alpha' = 0$  one has a geodesic). Then we have

$$\begin{aligned} N &= \cos \alpha \cos \beta e_1 + \cos \alpha \sin \beta e_2 - \sin \alpha e_3, \\ B &= -\sin \beta e_1 + \cos \beta e_2, \end{aligned}$$

and

$$\nabla_T N = (-\alpha' \sin \alpha \cos \beta - \frac{1}{2} \sin \beta) e_1 + (-\alpha' \sin \alpha \sin \beta + \frac{1}{2} \cos \beta) e_2 - \alpha' \cos \alpha e_3.$$

Now we make use of the second Frenet equation to obtain

$$-\tau = \langle \nabla_T N, B \rangle = \frac{1}{2}$$

□

Thus we have

**Corollary 4.2.** *Let  $\gamma : I \rightarrow \mathbb{H}_3$  be a non-geodesic biharmonic helix parametrized by arc length. Then*

$$(4.1) \quad \begin{cases} B_3 = \text{constant} \neq 0; \\ N_3 = 0; \\ k^2 + \tau^2 = \frac{1}{4} - B_3^2. \end{cases}$$

## 5. EXPLICIT FORMULAS FOR NON-GEODESIC BIHARMONIC CURVES IN $\mathbb{H}_3$

In this section we use the previous results to derive the explicit parametric equations of non-geodesic biharmonic curves in the Heisenberg group  $\mathbb{H}_3$ .

We first prove the following

**Lemma 5.1.** *Let  $\gamma : I \rightarrow \mathbb{H}_3$  be a non-geodesic curve parametrized by arc length. If  $N_3 = 0$ , then*

$$(5.1) \quad T(s) = \sin \alpha_0 \cos \beta(s) e_1 + \sin \alpha_0 \sin \beta(s) e_2 + \cos \alpha_0 e_3,$$

where  $\alpha_0 \in \mathbb{R}$ .

*Proof.* If  $\gamma' = T = T_1 e_1 + T_2 e_2 + T_3 e_3$ , and  $\|T\| = 1$ , from

$$\begin{aligned}\nabla_T T &= (T'_1 + T_2 T_3) e_1 + (T'_2 - T_1 T_3) e_2 + T'_3 e_3 \\ &= kN\end{aligned}$$

it follows that  $N_3 = 0$  if and only if  $T'_3 = 0$ , i.e. if and only if  $T_3 = \text{constant}$ . Since  $T_3 \in [0, 1]$ , the lemma follows.  $\square$

In order to find the integral curves of the field given by (5.1) that are biharmonic but non-geodesic, we must first determine the function  $\beta(s)$ . After which, a simple integration will give the wanted parametric equations. We shall prove

**Theorem 5.2.** *The parametric equations of all non-geodesic biharmonic curves  $\gamma$  of  $\mathbb{H}_3$  are*

$$(5.2) \quad \begin{cases} x(s) = \frac{1}{A} \sin \alpha_0 \sin(As + a) + b, \\ y(s) = -\frac{1}{A} \sin \alpha_0 \cos(As + a) + c, \\ z(s) = \left( \cos \alpha_0 + \frac{(\sin \alpha_0)^2}{2A} \right) s \\ \quad - \frac{b}{2A} \sin \alpha_0 \cos(As + a) - \frac{c}{2A} \sin \alpha_0 \sin(As + a) + d, \end{cases}$$

where  $A = \frac{\cos \alpha_0 \pm \sqrt{5(\cos \alpha_0)^2 - 4}}{2}$ ,  $\alpha_0 \in (0, \arccos \frac{2\sqrt{5}}{5}] \cup [\arccos(-\frac{2\sqrt{5}}{5}), \pi)$  and  $a, b, c, d \in \mathbb{R}$ .

*Proof.* We shall make use of the Frenet formulas (3.1), and we shall take into account Corollary 4.2 and Lemma 5.1.

The covariant derivative of the vector field  $T$  given by (5.1) is

$$\begin{aligned}\nabla_T T &= \sin \alpha_0 (\cos \alpha_0 - \beta') (\sin \beta e_1 - \cos \beta e_2) \\ &= kN,\end{aligned}$$

where  $k = |\sin \alpha_0 (\cos \alpha_0 - \beta')|$ .

Without loss of generality, we can always assume that  $\sin \alpha_0 (\cos \alpha_0 - \beta') > 0$ . Then we obtain

$$(5.3) \quad k = \sin \alpha_0 (\cos \alpha_0 - \beta')$$

and

$$N = \sin \beta e_1 - \cos \beta e_2.$$

Next we have

$$(5.4) \quad B = T \times N = \cos \beta \cos \alpha_0 e_1 + \sin \beta \cos \alpha_0 e_2 - \sin \alpha_0 e_3$$

and

$$\nabla_T N = \cos \beta (\beta' - \frac{1}{2} \cos \alpha_0) e_1 + \sin \beta (\beta' - \frac{1}{2} \cos \alpha_0) e_2 - \frac{1}{2} \sin \alpha_0 e_3.$$

It follows that the geodesic torsion  $\tau$  of  $\gamma$  is given by

$$(5.5) \quad -\tau = \langle \nabla_T N, B \rangle = (\cos \alpha_0) \beta' + \frac{1}{2} - (\cos \alpha_0)^2.$$

If  $\gamma$  is a curve with  $\gamma' = T$ , then this curve is non-geodesic and biharmonic if and only if

$$(5.6) \quad \begin{cases} \beta' = \text{constant}, \\ \beta' \neq \cos \alpha_0, \\ k^2 + \tau^2 = \frac{1}{4} - B_3^2. \end{cases}$$

From (5.3), (5.4), (5.5) and (5.6) we obtain

$$(\beta')^2 - (\cos \alpha_0) \beta' + 1 - (\cos \alpha_0)^2 = 0.$$

From this last equation we obtain

$$\beta' = \frac{\cos \alpha_0 \pm \sqrt{5(\cos \alpha_0)^2 - 4}}{2} = A,$$

with the condition  $(\cos \alpha_0)^2 \geq \frac{4}{5}$  for reality of solutions, and therefore  $\beta(s) = As + a$ , where  $a \in \mathbb{R}$ .

In order to find the explicit equations for  $\gamma(s) = (x(s), y(s), z(s))$ , we must integrate the system  $\frac{d\gamma}{ds} = T$ , that in our case is

$$\begin{cases} \frac{dx}{ds} = \sin \alpha_0 \cos(As + a), \\ \frac{dy}{ds} = \sin \alpha_0 \sin(As + a), \\ \frac{dz}{ds} = \cos \alpha_0 + \frac{1}{2} \sin \alpha_0 (\sin(As + a)x(s) - \cos(As + a)y(s)). \end{cases}$$

The integration is immediate and yields (5.2). □

**Remark 5.3.** Biharmonic curves (5.2) can be obtained by intersecting the two surfaces  $S$  and  $S'$  given by:

$$(5.7) \quad S(u, v) = \begin{cases} x(u, v) = \frac{1}{A} \sin \alpha_0 \sin(Au + a) + b, \\ y(u, v) = -\frac{1}{A} \sin \alpha_0 \cos(Au + a) + c, \\ z(u, v) = v, \end{cases}$$

and

$$(5.8) \quad S'(u, v) = \begin{cases} x'(u, v) = \frac{v}{A} \sin \alpha_0 \sin(Au + a) + b, \\ y'(u, v) = -\frac{v}{A} \sin \alpha_0 \cos(Au + a) + c, \\ z'(u, v) = (\cos \alpha_0 + \frac{\sin^2 \alpha_0}{2A})u + \frac{b}{2}y'(u, v) - \frac{c}{2}x'(u, v) + d. \end{cases}$$

The surface  $S$  has constant non zero mean curvature; it is the “round cylinder” with rulings parallel to the axis of revolution of  $\mathbb{H}_3$  at the point  $(b, c, 0)$  and as directrix the circle in the plane  $z = 0$  around this point; this circle has constant (geodesic) curvature also in  $\mathbb{H}_3$ . This cylinder has constant non zero mean curvature and zero Gaussian curvature also in  $\mathbb{H}_3$ . The surface  $S'$  is a “helicoid” which is minimal in the Heisenberg group  $\mathbb{H}_3$ , as one can easily check by using the formulas given by Bekkar in [Be]. Moreover, the biharmonic curves are geodesics of the cylinder and the cylinder is never a biharmonic surface. (With regard to the study of biharmonic surfaces, a paper devoted to invariant non-minimal biharmonic surfaces in  $\mathbb{H}_3$  is in preparation.)

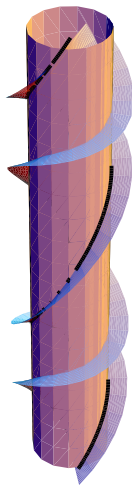


FIGURE 1.

Figure 1 shows this intersection and is obtained for  $a = b = c = 1$  and  $\sin \alpha_0 = \frac{1}{\sqrt{10}}$ .

In fact, the intersection of  $S$  and  $S'$  is the union of two curves. Figure 1 shows the biharmonic one.

**Remark 5.4.** At each point  $p \in \mathbb{H}_3$  the vectors tangent to biharmonic curves form a solid cone  $\mathcal{C}_p$  in  $T_p\mathbb{H}_3$ . For each point  $p \in \mathbb{H}_3$  and each vector  $X_p \in T_p\mathbb{H}_3 \setminus \mathcal{C}_p$ , the only biharmonic curve  $\gamma$  arising from  $p$  and such that  $\dot{\gamma}(p) = X_p$  is the geodesic determined by  $p$  and  $X_p$ . Thus, any  $X_p \in \mathcal{C}_p$  is simultaneously tangent to a geodesic and to a non-geodesic biharmonic curve. We visualize this fact in the following picture, where the geodesic is the curve not lying on the helicoid, while the interior edge is the biharmonic curve.

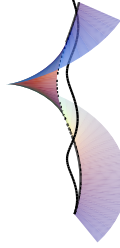


FIGURE 2.

**Remark 5.5.** Let's denote by  $\gamma$  the curve (5.2) when  $b = c = d = 0$ , and by  $\tilde{\gamma}$  the curve (5.2) when  $b^2 + c^2 > 0$ . Of course, both are helix in  $\mathbb{H}_3$ . Every biharmonic curve  $\tilde{\gamma}$  can be obtained from  $\gamma$  by means of a left-translation, i.e.  $\tilde{\gamma} = L_{(b,c,d)} \circ \gamma$ , and we note that  $\gamma$  is a helix in  $\mathbb{R}^3$ , but  $\tilde{\gamma}$  is not.

**Remark 5.6.** The vector field  $T$  tangent to a curve given by (5.2) is transverse to the contact structure of  $\mathbb{H}_3$  (for the contact geometry one can see, for example, [Go]) determined by the 1-form

$$\theta^3 = dz - \frac{xdy - ydx}{2}.$$

(It easy to verify that the contact condition,  $\theta^3 \wedge d\theta^3 \neq 0$ , is satisfied.) In fact one has

$$\theta^3(T) = \cos \alpha_0 \neq 0$$

for  $\alpha_0 \in (0, \arccos \frac{2\sqrt{5}}{5}] \cup [\arccos(-\frac{2\sqrt{5}}{5}), \pi)$ .

Consider now a curve  $\gamma : I \rightarrow \mathbb{H}_3$  tangent to the contact structure (such a curve is called a *Legendre curve*), parametrized by arc length. Its velocity vector field  $X$  has then the expression

$$X = \cos \psi(s)e_1 + \sin \psi(s)e_2.$$

It is not difficult to see that the vector fields  $T$  and  $X$  coincide for  $\alpha_0 = \frac{\pi}{2}$ . It follows

that the Legendre curves of the Heisenberg group  $H_3$  are biharmonic if and only if they are geodesic.

**Remark 5.7.** The one-parameter subgroups  $\sigma(u) = \exp uX$  are biharmonic if and only if they are geodesic. In fact, if  $\sigma(u) = \exp uX$  is not a geodesic, then  $k$  and  $\tau$  are always related by the formula (see [PS])

$$(5.9) \quad k^2 + \tau^2 = \frac{1}{4}, \quad \text{and} \quad B_3 \neq 0.$$

Now, the assertion follows from the second equation in (3.2).

**Remark 5.8.** Finally we note that the methods of this paper can be extended to study biharmonic curves in the Cartan-Vranceanu 3-manifolds, namely the Riemannian spaces  $(\mathbb{R}^3, ds_{m,l}^2)$ , where the Riemannian metrics  $ds_{m,l}^2$  are defined by

$$(5.10) \quad ds_{m,l}^2 = \frac{dx^2 + dy^2}{[1 + m(x^2 + y^2)]^2} + \left( dz + \frac{l}{2} \frac{ydx - xdy}{[1 + m(x^2 + y^2)]} \right)^2, \quad l, m \in \mathbb{R}.$$

This two-parameter family of metrics reduces to the metric (2.1) for  $m = 0$  and  $l = 1$  (see [Pi1], [Pi2], [Vr] and [Ca] for a discussion of these metrics and their properties).

The system for the non-geodesic biharmonic curves corresponding to the metric  $ds_{m,l}^2$  can be obtained by using the same techniques, and it turns out to be

$$(5.11) \quad \begin{cases} k = \text{constant} \neq 0; \\ k^2 + \tau^2 = \frac{l^2}{4} - (l^2 - 4m)B_3^2; \\ \tau' = (l^2 - 4m)N_3B_3. \end{cases}$$

For  $l^2 - 4m = 0$  the metric (5.10) has to be of constant curvature  $\frac{l^2}{4}$  and we have two cases

- (i) if  $l = 0$ , then  $m = 0$ , and we are in the Euclidean space, where  $\gamma(s)$  is biharmonic if and only if it is a line;
- (ii) if  $l \neq 0$ , the metric (5.10) has constant positive sectional curvature, and therefore the non-geodesic biharmonic curves are circles or spherical helices (see [CMO1]).

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